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LETTER TO THE EDITOR

Time-dependent real-space renormalisation group for the Ising system on a triangular lattice

Ya'akov Achiam†

Department of Mathematical Physics, University of Birmingham, Birmingham B15 2TT, UK

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Abstract. A real-space renormalisation group analysis of Glauber's equation of motion for a two-dimensional Ising system on a triangular lattice is carried out up to the second order of the cumulant approximation. The calculations yield two dynamic modes, and a value $z = 2{\cdot}2$ is found for the dynamic exponent.

In the last few years, progress has been made in the understanding of critical dynamics using the renormalisation group (RG) technique (Hohenberg and Halperin 1977). Most of the work on this subject is based on the ϵ expansion around four or six dimensions. Recently, results obtained by the real-space static RG technique (Niemeijer and van Leeuwen 1974, 1976, Kadanoff *et al* 1976) stimulated the generalisation of this method to the study of critical dynamics. Such a generalisation, which yields both the static and the dynamic properties of an Ising system was discussed by Achiam and Kosterlitz (1978). They used a first-order cumulant approximation to perform a time-dependent real-space renormalisation group (TDRS) analysis of the kinetic master equation proposed by Glauber (1963). They were able to confirm the existence of the dynamic scaling hypothesis (Halperin and Hohenberg 1969, Ferrel *et al* 1968) within the above approximation. They calculated the dynamic exponent z, which according to the dynamic hypothesis relates the time scale τ to the correlation length ξ, $\tau \sim \xi^z$ (Ma 1976a, Hohenberg and Halperin 1977). However, by comparison with other numerical works (see later), it is suspected that they found a value of z which is considerably higher than expected. The authors speculated that this fact is a result of the approximation which is known to give poor results for the magnetic exponent β (the method gives $\beta = -0{\cdot}15$ instead $+0{\cdot}125$) (Barber 1978).

In this Letter we want to report on a study of the critical dynamics of a two-dimensional Ising spin system on a triangular lattice using the TDRS suggested by Achiam and Kosterlitz (1978). We performed the cumulant approximation up to second order. For the kind of block-spin transformation that we chose, it has been suggested that the second-order cumulant approximation should be the optimal choice (Hemmer and Verlarde 1976). Because the values, $\nu = 0{\cdot}95$, $\beta = 0{\cdot}15$, predicted by this method for the static exponents are good, we expect that the value $z = 2{\cdot}16$ obtained for the dynamic exponent should be a good approximation to the exact

† Permanent address: Nuclear Research Centre, Negev, PO Box 9001, Beer-Sheva, Israel.

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value. This exponent, z, describes the decay of the slowest mode which is a combination of disturbances involving magnetic-like spin operators and a sum of product of three nearest-neighbour (NN) spins. We have also found a faster mode which is critically slowed down, with an exponent $z_2 = 1{\cdot}49$. Using the nomenclature of Wilson and Kogut (1974), we found that the main contribution to the slowest mode comes from the relevant odd perturbation, but also includes contributions from irrelevant odd operators. A similar situation appears in the one-dimensional Ising–Glauber model, where one cannot perform a direct separation into slow and fast modes (Achiam 1978). This problem does not arise in the ϵ expansion technique.

In the literature we have not found calculations applicable to the triangular lattice, except those based on the conventional theory. However, a few attempts have been made to study the dynamics of the two-dimensional Ising model on a square lattice using a variety of different methods. The values of z which these calculations suggest are:

(*a*) $z = 1{\cdot}85$ (Stoll *et al* 1973); $z = 2{\cdot}30$ (Bolton and Johnson 1976) by standard Monte-Carlo methods;
(*b*) $z = 1{\cdot}4$ (Ma 1976b) by RG Monte-Carlo methods;
(*c*) $z = 2{\cdot}0$ (Yahata and Suzuki 1969) by high-temperature expansion;
(*d*) $z = 2{\cdot}7$ (Achiam and Kosterlitz 1978); $z = 1{\cdot}6$, $z = 1{\cdot}77$ (Kinzel 1978) by TDRS (first-order) cumulant approximation;
(*e*) $z = 2{\cdot}16$ (present work) TDRS in the second order of the cumulant approximation (triangular lattice).

In all the above calculations, except in those by Ma (1976b), Achiam and Kosterlitz (1978) and the present work, the static and the dynamic properties were not calculated simultaneously.

In the following we shall describe briefly the model and the basic ideas of the TDRS. The reader is referred to Niemeijer and van Leeuwen (1976) for a general discussion of the static real-space RG. We shall not enter into the details of the calculation but simply describe the essential parts of the method and stress the differences between the first-order cumulant approximation in which only one mode exists, and the second-order aproximation which yields two modes.

The equilibrium properties of the two-dimensional Ising model are determined by a reduced Hamiltonian. $\bar{H} = -H/kT = \Sigma_a K_a S_a(\sigma)$ where S_a are extensive functions of the spins, $\sigma = \pm 1$, and K_a are the conjugate fields. The three relevant even operators in the second order of the cumulant approximation are the sum of first NN (denoted by S_1), next NN (denoted by S_2) and third NN (denoted by S_3) (Niemeijer and van Leeuwen 1974). We shall consider small deviations from equilibrium, which are described by

$$\phi(\sigma, t) \equiv P(\sigma, t)/P_e(\sigma) = 1 + \sum_i O_i(\sigma) h_i(t) \equiv 1 + (\boldsymbol{O}(\sigma) \,.\, \boldsymbol{h}(t)),$$

where $P_e \equiv \exp(\{\bar{H}\}/Z)$ and $P(\sigma, t)$ are the equilibrium and the time-dependent probability distributions of the spins, respectively; the $Q_i(\sigma)$ are extensive functions of the spins, odd under spin reversal, and h_i are the corresponding time-dependent effective fields. We shall denote the magnetisation by O_1. The functions O_2, O_3 and O_4 are sum of products of three NN spins. In O_2 all the spins are NN, in O_3 two of them are next NN and in O_2, two of them are third NN. In the model that we are studying, the approach towards equilibrium is via interaction with a heat bath and governed by the kinetic equation of Glauber (1963);

$$\tau\, \mathrm{d}P(\sigma, t)/\mathrm{d}t = -P_e(\sigma)\sum_i W_i(\sigma_i)(1-p_i)\phi(\sigma, t) \equiv -\sum_i \mathscr{L}_i\phi(\sigma, t) \tag{1}$$

where τ is a time scale describing the effective interaction with the heat bath, and the operator p_i flips the spin σ_i: $p_i f(\sigma_i) = f(-\sigma_i)$. The transition probabilities $W_j(\sigma_j)$ are subject to the condition of detailed balance which ensures that $P(\sigma, \infty) = P_e(\sigma)$. A convenient choice for W is,

$$W_j(\sigma_j) = (P_e(-\sigma_j)/P_e(\sigma_j))^{1/2}. \tag{2}$$

It can be shown that different forms of W_j only lead to the appearance of dynamic transients, hence the particular form (2) does not restrict the generality of the model.

The RG transformation is carried out exactly, as in equilibrium, using a cell-spin transformation $T(\mu, \sigma)$ (Niemeijer and van Leeuwen 1976),

$$P(\mu, t) = \sum_\sigma T(\mu, \sigma)P(\sigma, t). \tag{3}$$

In equation (3) time plays the role of a parameter. $P(\sigma, t)$ can be represented by a reduced Hamiltonian, $P(\sigma, t) \equiv \exp(\{\bar{H}(\sigma, t)\})/Z$. The time-dependent parameters of $H(\sigma, t)$ are transformed under the RG in the same way as the static ones. Let us examine the behaviour of the kinetic equation (1) under space scaling. Equation (1) can be written as,

$$\tau\frac{\mathrm{d}}{\mathrm{d}t}[P_e(\sigma)(\boldsymbol{O}(\sigma)\cdot\boldsymbol{h}(t))] = -\sum_i \mathscr{L}_i(\sigma)(\boldsymbol{O}(\sigma)\cdot\boldsymbol{h}(t)). \tag{4}$$

By operating on both sides of (4) with $\Sigma_\sigma T(\mu, \sigma)$ we get:

$$t\frac{\mathrm{d}}{\mathrm{d}t}[P'_e(\mu)(\boldsymbol{O}(\mu)\cdot\boldsymbol{\Lambda h}(t))] = -\sum_i \mathscr{L}'_i(\mu)(\boldsymbol{O}(\mu)\cdot\boldsymbol{\Omega h}(t)) \tag{5}$$

where $\boldsymbol{\Lambda}$ and $\boldsymbol{\Omega}$ are matrices which are defined by the transformation and the primes denote quantities with renormalised interactions. Using the definition $\tilde{\boldsymbol{h}}(t) = \boldsymbol{\Omega h}(t)$, equation (5) becomes:

$$\tau\frac{\mathrm{d}}{\mathrm{d}t}[P'_e(\mu)(\boldsymbol{O}(\mu)\cdot\boldsymbol{\Lambda\Omega}^{-1}\boldsymbol{h}(t))] = -\sum_i \mathscr{L}'_i(\mu)(\boldsymbol{O}(\mu)\cdot\tilde{\boldsymbol{h}}(t)). \tag{6}$$

The matrix $\boldsymbol{\Lambda\Omega}^{-1}$ can be diagonalised using right and left eigenvectors corresponding to a set of eigenvalues λ_i. These eigenvectors describe the dynamic normal modes of the system. If $\tilde{\boldsymbol{h}}(t)$ is the jth eigenvector, then after a time rescaling $\tau' = \lambda_j\tau$, equation (6) will have the same form as equation (4). The transformation $T(\mu, \sigma)$ together with the rescaling of time constitutes the TDRS. Simple scaling arguments (Ma 1976a) connect λ_i to z_i, the corresponding dynamic exponent, by $\lambda_i = b^{z_i}$, where b is the space scale factor. They also guarantee the existence of the dynamic scaling hypothesis. The largest z_i defines the slowest mode.

The results reported previously were obtained using the block-spin transformation,

$$T = \prod_\alpha [1 + \tfrac{1}{2}\mu_\alpha(\sigma_1 + \sigma_2 + \sigma_3 - \sigma_1\sigma_2\sigma_3)]/2,$$

where the cell α includes the spins σ_1, σ_2 and σ_3, and each spin belongs to one and only one cell which has a block-spin variable $\mu_\alpha = \pm 1$. This block-spin transformation was suggested by Niemeijer and van Leeuwen (1974) and was shown to give a lowest upper bound for the equilibrium free energy by Barber (1978). The renormalisation of the left-hand side of (4) which defined Λ in (5) is the straightforward RG of static odd perturbation as described by Niemeijer and van Leeuwen (1974). The trace over $\{\sigma\}$ is carried out in the cumulant approximation, where the intercell interactions are taken as a perturbation, V, on the intracell interactions. The magnetic perturbation, O_1, under the RG creates interactions of the form O_2, O_3 and O_4. Hence these operators must be included *a priori* in the parameter space.

The renormalisation of the right-hand side is more complicated. The operator $\mathscr{L}_i$ contains the factor $P_e(\sigma)W_i(\sigma_i)$ which is independent of σ_i. Hence the trace over σ_i can be carried out exactly. Using symmetry arguments, discussed by Achiam and Kosterlitz (1978), one can see that the $\mathscr{L}_i(\sigma_i)$ is renormalised to an operator $\mathscr{L}'_\alpha(\mu_\alpha)$ which does not include interactions with the block-spin μ_α, $\sigma_i \subset \alpha$. Thus $\mathscr{L}'_\alpha(\mu_\alpha) \sim P'_e(\mu)W_\alpha(\mu_\alpha)$. However, by the same arguments one can see that second-order terms which are created by interactions of two spins via an intermediate spin, belong to the cell α, have a different form to the corresponding terms in $P'_e(\mu)$. This is equivalent to the appearance of perturbations O_2, O_3 and O_4, which have to be included in ϕ, as was mentioned earlier.

The above approximation scheme yields two 4×4 matrices, $\boldsymbol{\Lambda}$ and $\boldsymbol{\Omega}$. Throughout the calculation we have assumed that h_2, h_3 and h_4 are $O(V^2)$. This artificial assumption leads to a finite, self-consistent parameter space of the odd parameters. Similar arguments are used in the even parameter space in order to create a finite parameter space. This procedure contains some anomalies, e.g. the static RG contains a negative eigenvalue (Niemeijer and van Leeuwen 1974), and the matrices $\boldsymbol{\Lambda}$ and $\boldsymbol{\Omega}$ are almost singular. To overcome these difficulties we treated h_3 and h_4 as dependent variables, and represented them to $O(V^2)$ in terms of h_1. Thus $\boldsymbol{\Lambda}$ and $\boldsymbol{\Omega}$ reduced to 2×2 matrices, $\tilde{\boldsymbol{\Lambda}}$ and $\tilde{\boldsymbol{\Omega}}$, respectively. At the static fixed point, they have the values:

$$\tilde{\boldsymbol{\Lambda}} = \begin{bmatrix} 2\cdot 7721 & 0\cdot 6894 \\ -0\cdot 0377 & 0\cdot 3001 \end{bmatrix} \qquad \tilde{\boldsymbol{\Omega}} = \begin{bmatrix} 0\cdot 8293 & -0\cdot 6554 \\ -0\cdot 0120 & 0\cdot 1481 \end{bmatrix}.$$

The non-commutativity of $\tilde{\boldsymbol{\Lambda}}$ and $\tilde{\boldsymbol{\Omega}}$ is due to the memory effects. The long-time behaviour is determined by the multiplication of the eigenvalues of each matrix. This leads to the following asymptotic eigenvalues and dynamic exponents: $\lambda_1 = 3\cdot 285$, $z_1 = 2\cdot 165$; $\lambda_2 = 2\cdot 272$, $z_2 = 1\cdot 49$.

The following features of the cumulant approximation should be mentioned: (*a*) the transformation T we are using does not preserve the symmetry of the lattice. This strongly affects the calculations involving odd perturbations which are essential to our study. The symmetry must be restored after each renormalisation. (*b*) The nature of the convergence of the cumulant approximation is not clear. However, the second-order approximation is assumed to be the optimal one for the static limit in the above spin-cell transformation. Hence we hope that it gives similar accuracy in the above calculation. (*c*) The main advantage of the cumulant approximation is the elimination of boundary effects which affect the dynamic behaviour more strongly than they do to the equilibrium behaviour. We conclude by remarking that the examination of the asymptotic solution of (1) reveals terms in ϕ, of order V which have special symmetry. These terms, which have not been discussed here are scaled with λ_1 and do not affect the above discussion.

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